MATH 2050C Lecture on 4/24/2020

Note: NO CLASS NEXT FRIDAY. Review lecture on WED.

Common Mistake: Let $f : A \longrightarrow \mathbb{R}$.

$\lim_{x \to c} f(x) = L$ VS f is cts at c

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.	$\forall \varepsilon > 0, \exists \delta > 0$ s.t.
$\lvert f(x) - L \rvert < \varepsilon$	$\lvert f(x) - f(c) \rvert < \varepsilon$
whenever $x \in A$	whenever $x \in A$
and $0 < \lvert x - c \rvert < \delta$	and $\lvert x - c \rvert < \delta$
① c needs to be a cluster pt. ($c \in A$ OR $c \notin A$)	① c needs NOT be a cluster pt. BUT $c \in A$.
② $f(c)$ may not be defined	② $f(c)$ has to be defined
③ We don't care the case $x = c$.	③ We CARE about $x = c$.

Def^{n}: $f : A \to \mathbb{R}$ is uniformly cts (on A) or $B \subseteq A$

iff $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$\lvert f(u) - f(v) \rvert < \varepsilon \quad \text{whenever } u, v \in A \text{ (or } B\text{)}, \ \lvert u - v \rvert < \delta.$$

($\Leftrightarrow$ "f is cts at every $c \in A$ with the same δ for ALL c.")

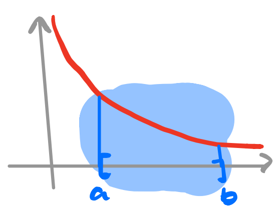
Q1: When is $f : A \to \mathbb{R}$ NOT unif. cts?

Non-uniform Continuity Criteria:

$f : A \to \mathbb{R}$ is NOT uniformly cts (on A) $\Leftrightarrow$ $\exists \varepsilon_0 > 0$ and seq. (u_n), (v_n) in A s.t. $\lvert u_n - v_n \rvert < \frac{1}{n}$ BUT $\lvert f(u_n) - f(v_n) \rvert \geq \varepsilon_0$ $\forall n \in \mathbb{N}$

E.g.) $f(x) = \frac{1}{x}$ is NOT unif. cts on $(0, \infty)$.

BUT is cts on $(0, \infty)$.



Note: $f(x) = \frac{1}{x}$ IS unif. cts on $[a, b]$.

(by Thm A below)

Q2 : When is $f: A \to \mathbb{R}$ unif. cts. ? [assume : A is an interval.]

Thm A : (Uniform Continuity Thm.)

Let $f: [a,b] \to \mathbb{R}$ be a function defined on a closed & bdd interval.

$$f \text{ is cts on } [a,b] \implies f \text{ is unif. cts on } [a,b]$$

$\Longleftarrow$ (always true for A)

Proof: By Contradiction.

Suppose NOT, i.e. f is cts BUT NOT uniformly cts on $[a,b]$.

- By Non-uniform continuity criteria.

$\exists \, \varepsilon_0 > 0$, seq. (u_n), (v_n) in $[a,b]$ s.t.

(*) —— $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$.

- By Bolzano-Weierstrass. $\exists$ convergent subseq. (u_{n_k}) of (u_n).

say $\lim_{k\to\infty} (u_{n_k}) =: u^* \in [a,b]$

same!

- Claim : $\lim_{k\to\infty} (v_{n_k}) = u^*$

Pf of Claim: By (*), $|u_{n_k} - v_{n_k}| < \frac{1}{n_k} \quad \forall k \in \mathbb{N}$

i.e. $u_{n_k} - \frac{1}{n_k} < v_{n_k} < u_{n_k} + \frac{1}{n_k} \quad \forall k \in \mathbb{N}$

Take $k \to \infty$, $n_k \to \infty$, by Squeeze Thm for seq., $\lim_{k\to\infty} (v_{n_k}) = u^*$. Claim □

- Notice by (*),

u^*

$$|f(u_{n_k}) - f(v_{n_k})| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}$$

As f is cts on $[a,b]$, in particular, at $x = u^*$, take $k \to \infty$ above

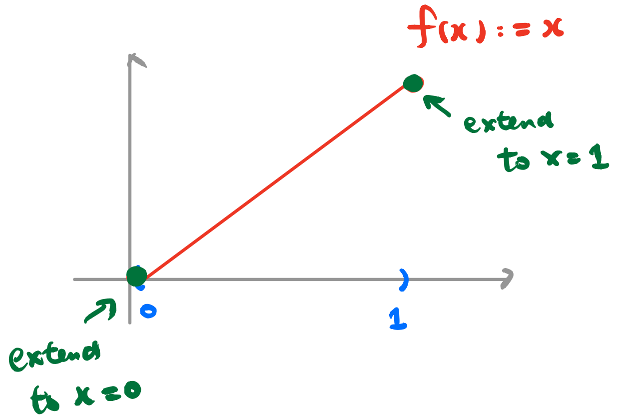
$$0 = |f(u^*) - f(u^*)| = \lim_{k\to\infty} |f(u_{n_k}) - f(v_{n_k})| \geq \varepsilon_0 > 0$$

Contradiction !! □

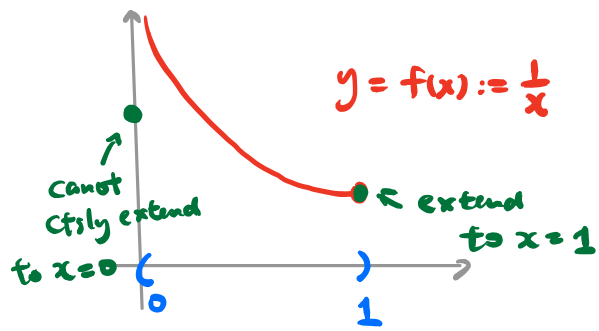
Q: Given a cts $f : (a,b) \to \mathbb{R}$.

can we extend it to a function $\bar{f} : [a,b] \to \mathbb{R}$ st. $\bar{f}$ is cts ?

E.g.) Yes.



E.g.) No.



Thm B: (Continuous Extension Thm)

If $f : (a,b) \to \mathbb{R}$ is uniformly cts on (a,b). (open interval)

then $\exists$ "extension" $\bar{f} : [a,b] \to \mathbb{R}$ st. (closed interval)

(i) $\bar{f}(x) = f(x) \quad \forall x \in (a,b)$.

and (ii) $\bar{f}$ is cts on $[a,b]$.

Remark: (a) By Thm A, $\bar{f}$ is unif. cts on $[a,b]$.

(b) Such an extension $\bar{f}$ is unique.

Lemma: Let $f : A \to \mathbb{R}$ be uniformly cts on A. Then,

(x_n) Cauchy seq. in A $\Longrightarrow$ $(f(x_n))$ Cauchy seq. in $\mathbb{R}$

(Caution: $\lim (x_n)$ may not be in A)

[i.e. unif. cts function preserves Cauchy seq.]

Remark: f ONLY cts on A. Then, by Seq. Criteria.

(x_n) convergent seq. in A with $\lim (x_n) = x^* \in A$ $\Longrightarrow$ $(f(x_n))$ convergent w/. $\lim(f(x_n)) = f(x^*)$.

Proof of Lemma:

Let $\varepsilon > 0$.

- By uniform continuity of f, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

(**) —— $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in A$, $|u - v| < \delta$

- Since (x_n) is Cauchy (seq. in A), for this $\delta > 0$, $\exists\ H = H(\delta) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \delta \quad \forall\, n, m \geq H$$

By (**), $|f(x_n) - f(x_m)| < \varepsilon \quad \forall\, n, m \geq H$

So, $(f(x_n))$ is Cauchy. ∎

Proof of Thm B:

Claim: $\lim_{x \to a} f(x)$ exists.

Pf of Claim: By seq. criteria, it suffices to show:

$\exists\ L \in \mathbb{R}$ s.t. for any seq. (x_n) in (a, b)

s.t. $\lim(x_n) = a$, we have $\lim(f(x_n)) = L$

same L for ALL seq (x_n)

Step 1: Find one such L.

Choose a seq. $(x_n) := \left(a + \frac{1}{n}\right)$ $\forall\, n \in \mathbb{N}$.

[Note: $x_n \in (a, b)$ $\forall\, n$ large.]

Since $(x_n) \to a$, it's Cauchy.

By Lemma, $(f(x_n))$ is Cauchy

By Cauchy Criteria, $\exists\ L \in \mathbb{R}$ st. $\lim(f(x_n)) = L$.

Caution: This L may not work for other seq $(x_n') \to a$.

Given $f : (a, b) \to \mathbb{R}$ cts.

if such an extension

$\bar{f} : [a, b] \to \mathbb{R}$ exists:

then

$\bar{f}(a) = \lim_{x \to a} \bar{f}(x) \overset{\text{define}}{=} \lim_{x \to a} f(x)$

$\bar{f}(b) = \lim_{x \to b} \bar{f}(x) \overset{\text{define}}{=} \lim_{x \to b} f(x)$

exists?

Step 2: Show this L works for ALL seq. $(x_n') \to a$.

Take any seq. (x_n') in (a,b) s.t. $\lim(x_n') = a$

By the same argument in Step 1, $\lim(f(x_n')) = L'$ for some $L' \in \mathbb{R}$

We want to show $L' = L$.

Note: $\lim |x_n - x_n'| = |a-a| = 0$

Fix $\varepsilon > 0$.

- Since f is unif. cts on (a,b), $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

(***) —— $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in (a,b)$, $|u-v| < \delta$

- Since $\lim |x_n - x_n'| = 0$, $\exists\, K = K(\delta) \in \mathbb{N}$ s.t

$$|x_n - x_n'| < \delta \qquad \forall\, n \geq K$$

By (***), $|f(x_n) - f(x_n')| < \varepsilon \quad \forall\, n \geq K$.

Take $n \to \infty$, since $\lim f(x_n) = L$ and $\lim f(x_n') = L'$

$$\Rightarrow \qquad |L - L'| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, so $L = L'$.

Define:

$$\bar{f}(x) = \begin{cases} f(x) & , \quad x \in (a,b) \\ \lim\limits_{x \to a} f(x) & \text{at } x = a \\ \lim\limits_{x \to b} f(x) & \text{at } x = b \end{cases}$$

Clearly, $\bar{f}$ is well-defined and cts on $[a,b]$.

$\square$

Example: $f(x) = \sin\frac{1}{x}$ on $(0,1)$

Example: $f(x) = x \sin\frac{1}{x}$ on $(0,1)$